(4)

Systems
Optimization
Laboratory

Fixed Points by Ishikawa Iterations

by
Jen-Chih Yao

TECHNICAL REPORT SOL 89-19

December 1989

Department of Operations Research
Stanford University
Stanford, CA 94305

(2)

SYSTEMS OPTIMIZATION LABORATORY
DEPARTMENT OF OPERATIONS RESEARCH
STANFORD UNIVERSITY
STANFORD, CALIFORNIA 94305-4022

Fixed Points by Ishikawa Iterations

by
Jen-Chih Yao

TECHNICAL REPORT SOL 89-19

December 1989

Research and reproduction of this report were partially supported by the National Science Foundation grant DMS 8913089; U.S. Department of Energy grant DE-FG03-87-ER-25028, and the Office of Naval Research grant N00014-89-J-1659.

FIXED POINTS BY ISHIKAWA ITERATIONS

JEN-CHIH YAO

ABSTRACT. In this paper, we introduce a class of mappings called generalized quasi-nonexpansive mappings in a Hilbert space. It is shown that a certain Ishikawa iterative process generated by a continuous generalized quasi-nonexpansive and monotone mapping on a compact and convex subset of a Hilbert space always converges strongly to a fixed point of the mapping without any precondition.

1. INTRODUCTION

In [1], Ishikawa has shown that a certain mean value sequence generated by a Lipshitzian and pseudo-contractive mapping on a compact and convex subset of a Hilbert space with arbitrary chosen initial point converges strongly to a fixed point of the mapping. In this paper, we introduce a class of mappings called generalized quasi-nonexpansive mappings on a Hilbert space and show that a certain Ishikawa iterative process generated by a continuous generalized quasi-nonexpansive and monotone mapping on a compact and convex subset of a Hilbert space always converges strongly to a fixed point of the mapping without any precondition.

2. PRELIMINARIES

Let K be a nonempty subset of a Hilbert space X. A mapping T from K into itself is said to be *generalized quasi-nonexpansive* on K if T has a fixed point in K and for any fixed point p of T in K, we have

$$\|T(x)-p\|^2 \leq \alpha\|x-p\|^2 + \beta\|T(x)-x\|^2$$

for all $x \in K$, where $\alpha, \beta \geq 0$ with $\alpha + 2\beta \leq 1$. We note that there exists a generalized quasi-nonexpansive mapping which is not pseudo-contractive. For example, let T be a mapping from $[0, 2/3]$ into itself defined by $T(x) = x^2$ for all $x \in [0, 2/3]$. Then $x = 0$ is the only fixed point of T and it is easy to see that T is generalized quasi-nonexpansive on $[0, 2/3]$. But T is not pseudo-contractive because if we let $x = 2/3$ and $y = 1/2$, then

$$\|T(x) - T(y)\|^2 > \|x - y\|^2 + \|(I - T)(x) - (I - T)(y)\|^2.$$

Given $x_1 \in K$ and sequences of real numbers $\{\alpha_n\}$ and $\{\beta_n\}$ with $0 \leq \alpha_n, \beta_n \leq 1$, and $\sum_{n=1}^{\infty} \alpha_n \beta_n (1 - \beta_n) = \infty$, let $I(x_1, \alpha_n, \beta_n, T)$ be a sequence $\{x_n\}_{n=1}^{\infty}$ defined iteratively by

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T[\beta_n T(x_n) + (1 - \beta_n)x_n].$$

Ishikawa [1] introduced this iteration scheme by imposing the following different restrictions on α_n and β_n: $0 \leq \alpha_n \leq \beta_n \leq 1$ for all n, $\lim_{n \to \infty} \beta_n = 0$, and $\sum_{n=1}^{\infty} \alpha_n \beta_n = \infty$.

3. The Main Result

We now state and prove the main result of this paper.

THEOREM 1. *Let K be a nonempty compact and convex subset of a Hilbert space X and T be a continuous generalized quasi-nonexpansive and monotone mapping from K into itself. Then for arbitrary $x_1 \in K$, the sequence $I(x_1, \alpha_n, \beta_n, T)$ converges strongly to a fixed point of T.*

PROOF. Since T is monotone, $\langle T(x) - T(y), x - y \rangle \geq 0$ for all $x, y \in K$. Also since T is generalized quasi-nonexpansive, for any fixed point $p \in K$, we have for any $x \in K$,

$$\begin{aligned}
\|T(x) - p\|^2 &= \|T(x) - T(p)\|^2 \\
&\leq \alpha\|x - p\|^2 + \beta\|T(x) - x\|^2 \\
&= \alpha\|x - p\|^2 + \beta(\|T(x) - p\|^2 + \|x - p\|^2 - 2\langle T(x) - p, x - p \rangle) \\
&\leq (\alpha + \beta)\|x - p\|^2 + \beta\|T(x) - p\|^2.
\end{aligned}$$

Hence for all $x \in K$,

$$\|T(x) - p\| \leq [(\alpha + \beta)/(1 - \beta)]^{1/2}\|x - p\| \leq \|x - p\|. \tag{1}$$

Since X is a Hilbert space, it can be shown that for all $x, y, z \in X$ and $0 \leq \lambda \leq 1$,

$$\|\lambda x + (1 - \lambda)y - z\|^2 = \lambda\|x - z\|^2 + (1 - \lambda)\|y - z\|^2 - \lambda(1 - \lambda)\|x - y\|^2. \tag{2}$$

If $x_1 = p$, then we are done. So we may assume that $x_1 \neq p$. Then by (1) and (2), we have

$$\begin{aligned}
\|x_{n+1} - p\|^2 &= (1 - \alpha_n)\|x_n - p\|^2 + \alpha_n\|T[\beta_n T(x_n) + (1 - \beta_n)x_n] - p\|^2 - \\
&\quad \alpha_n(1 - \alpha_n)\|T[\beta_n T(x_n) + (1 - \beta_n)x_n] - x_n\|^2 \\
&\leq (1 - \alpha_n)\|x_n - p\|^2 + \alpha_n\|\beta_n T(x_n) + (1 - \beta_n)x_n - p\|^2 \\
&= (1 - \alpha_n)\|x_n - p\|^2 + \alpha_n\beta_n\|T(x_n) - p\|^2 + \alpha_n(1 - \beta_n)\|x_n - p\|^2 - \\
&\quad \alpha_n\beta_n(1 - \beta_n)\|T(x_n) - x_n\|^2 \\
&\leq (1 - \alpha_n\beta_n)\|x_n - p\|^2 + \alpha_n\beta_n\|x_n - p\|^2 - \alpha_n\beta_n(1 - \beta_n)\|T(x_n) - x_n\|^2 \\
&= \|x_n - p\|^2 - \alpha_n\beta_n(1 - \beta_n)\|T(x_n) - x_n\|^2.
\end{aligned}$$

Hence for every positive integer m, $\|x_{m+1} - p\| \leq \|x_m - p\|$ and

$$\|x_{m+1} - p\|^2 \leq \|x_1 - p\|^2 - \sum_{n=1}^{m} \alpha_n\beta_n(1 - \beta_n)\|T(x_n) - x_n\|^2$$

from which it follows that $\sum_{n=1}^{\infty} \alpha_n\beta_n(1 - \beta_n)\|T(x_n) - x_n\|^2 < \infty$. Therefore, from the hypothesis $\sum_{n=1}^{\infty} \alpha_n\beta_n(1 - \beta_n) = \infty$, we have

$$\lim_{n \longrightarrow \infty} \|T(x_n) - x_n\| = 0. \tag{3}$$

Since K is compact, the sequence $\{x_n\}$ contains a convergent subsequence $\{x_{n_i}\}$ with limit $q \in K$. From (3) it follows that q is a fixed point of T. Now, for any $\epsilon > 0$, there exists an integer n_k so that $\|x_{n_k} - q\| \leq \epsilon$. Thus for all $n \geq n_k$, we have $\|x_n - q\| \leq \|x_{n_k} - q\| \leq \epsilon$. Consequently, the entire sequence $\{x_n\}$ also converges to q, and the result follows. □

In the case that the Hilbert space X in Theorem 1 is finite-dimensional, the compactness of the set K can be eliminated.

THEOREM 2. *Let K be a nonempty closed and convex subset of a finite-dimensional Hilbert space X and T be a continuous generalized quasi-nonexpansive and monotone mapping from K into itself. Then for arbitrary $x_1 \in K$, the sequence $I(x_1, \alpha_n, \beta_n, T)$ converges to a fixed point of T.*

PROOF. Let p be any fixed point of T and let B be the closed ball of X with center p and radius $\|x_1 - p\|$. Then $x_n \in B$ for all n. Since B is compact, the sequence $\{x_n\}$ contains a convergent subsequence $\{x_{n_i}\}$ with limit $q \in B$. By the same argument as that in the proof of Theorem 1, it can be shown that q is a fixed point of T, and hence the result follows. □

We note that the result of Theorem 2 may not be true if the Hilbert space X is infinite-dimensional. This is because that in this case, the set B in the proof of Theorem 2 is no longer compact.

REFERENCES

1. S. Ishikawa, *Fixed points by a new iteration method,* Proc. Amer. Math. Soc. **44** (1974), 147-150.

Department of Operations Research, Stanford University, Stanford, California 94305-4022

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER SOL 89-19	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) Fixed Points by Iterations		5. TYPE OF REPORT & PERIOD COVERED Technical Report
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) Jen-Chih Yao		8. CONTRACT OR GRANT NUMBER(s) N00014-89-J-1659
9. PERFORMING ORGANIZATION NAME AND ADDRESS Department of Operations Research - SOL Stanford University Stanford, CA 94305-4022		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS 1111MA
11. CONTROLLING OFFICE NAME AND ADDRESS Office of Naval Research - Dept. of the Navy 800 N. Quincy Street Arlington, VA 22217		12. REPORT DATE December 1989
		13. NUMBER OF PAGES 4 pp.
		15. SECURITY CLASS. (of this report) UNCLASSIFIED
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

This document has been approved for public release and sale; its distribution is unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Hilbert space; generalized quasi-nonexpansive mapping; monotone mapping; Ishikawa iterative process; fixed point.

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

Please see reverse side...

DD FORM 1 JAN 73 1473 EDITION OF 1 NOV 65 IS OBSOLETE

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

FIXED POINTS BY ISHIKAWA ITERATIONS

JEN-CHIH YAO

ABSTRACT. In this paper, we introduce a class of mappings called generalized quasi-nonexpansive mappings in a Hilbert space. It is shown that a certain Ishikawa iterative process generated by a continuous generalized quasi-nonexpansive and monotone mapping on a compact and convex subset of a Hilbert space always converges strongly to a fixed point of the mapping without any precondition.